ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 339 (2008) 1065–1072
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Equi-statistical convergence of positive linear operators

Sevda Karakuş [a], Kamil Demirci [a,*], Oktay Duman [b]

[a] *Ondokuz Mayıs University, Faculty of Sciences and Arts Sinop, Department of Mathematics, 57000 Sinop, Turkey*
[b] *TOBB Economics and Technology University, Faculty of Arts and Sciences, Department of Mathematics, Söğütözü 06530, Ankara, Turkey*

Received 22 March 2007
Available online 28 July 2007
Submitted by Richard M. Aron

Abstract

Balcerzak, Dems and Komisarski [M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007) 715–729] have recently introduced the notion of equi-statistical convergence which is stronger than the statistical uniform convergence. In this paper we study its use in the Korovkin-type approximation theory. Then, we construct an example such that our new approximation result works but its classical and statistical cases do not work. We also compute the rates of equi-statistical convergence of sequences of positive linear operators. Furthermore, we obtain a Voronovskaya-type theorem in the equi-statistical sense for a sequence of positive linear operators constructed by means of the Bernstein polynomials.

Keywords: Statistical convergence; Equi-statistical convergence; Korovkin-type approximation theorem; Bernstein polynomials; Voronovskaya-type theorem; Modulus of continuity

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence by Fast [10]. Recently various kinds of statistical convergence for sequences of functions have been introduced by Balcerzak et al. [2] (see also [6]). Especially, in [2], a kind of convergence lying between pointwise and uniform statistical convergence is presented. We first recall these convergence methods.

Let f and f_n belong to $C(X)$, which is the space of all continuous real valued functions on a compact subset X of the real numbers. Throughout the paper, we use the following notation:

$$\Psi_n(x,\varepsilon) := \left|\left\{k \leqslant n\colon \left|f_k(x) - f(x)\right| \geqslant \varepsilon\right\}\right| \quad (x \in X) \tag{1.1}$$

and

$$\Phi_n(\varepsilon) := \left|\left\{k \leqslant n\colon \|f_k - f\|_{C(X)} \geqslant \varepsilon\right\}\right|,$$

* Corresponding author. Present address: Sinop University, Faculty of Arts and Sciences, Department of Mathematics, 57000 Sinop, Turkey.
E-mail addresses: skarakus@omu.edu.tr (S. Karakuş), kamild@omu.edu.tr (K. Demirci), oduman@etu.edu.tr (O. Duman).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.07.050

where $\varepsilon > 0$, $n \in \mathbb{N}$, the symbol $|B|$ denotes the cardinality of the set B; and $\|f\|_{C(X)}$ denotes the usual supremum norm of f in $C(X)$. Then, following [2] we have the next definitions.

Definition 1.1. (f_n) is said to be statistically pointwise convergent to f on X if $\text{st-lim}_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$, i.e., for every $\varepsilon > 0$ and for each $x \in X$, $\lim_{n\to\infty} \frac{\Psi_n(x,\varepsilon)}{n} = 0$. Then, it is denoted by $f_n \to f$ (stat) on X.

Definition 1.2. (f_n) is said to be equi-statistically convergent to f on X if for every $\varepsilon > 0$, $\lim_{n\to\infty} \frac{\Psi_n(x,\varepsilon)}{n} = 0$ uniformly with respect to $x \in X$, which means that $\lim_{n\to\infty} \frac{\|\Psi_n(\cdot,\varepsilon)\|_{C(X)}}{n} = 0$ for every $\varepsilon > 0$. In this case, we denote this limit by $f_n \to f$ (equi-stat) on X.

Definition 1.3. (f_n) is said to be statistically uniform convergent to f on X if $\text{st-lim}_{n\to\infty} \|f_n - f\|_{C(X)} = 0$, or $\lim_{n\to\infty} \frac{\Phi_n(\varepsilon)}{n} = 0$. This limit is denoted by $f_n \rightrightarrows f$ (stat) on X.

Using the above definitions, the next result follows immediately.

Lemma 1.1. *$f_n \rightrightarrows f$ on X (in the ordinary sense) implies $f_n \rightrightarrows f$ (stat) on X, which also implies $f_n \to f$ (equi-stat) on X. Furthermore, $f_n \to f$ (equi-stat) on X implies $f_n \to f$ (stat) on X; and $f_n \to f$ on X (in the ordinary sense) implies $f_n \to f$ (stat) on X.*

However, one can construct an example which guarantees that the converses of Lemma 1.1 are not always true. Such an example was given in [2] as follows:

Example. For each $n \in \mathbb{N}$, define $g_n \in C[0,1]$ by

$$g_n(x) = \begin{cases} 2^{n+1}(x - \frac{1}{2^n}), & \text{if } x \in [\frac{1}{2^n}, \frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}], \\ -2^{n+1}(x - \frac{1}{2^{n-1}}), & \text{if } x \in [\frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}], \\ 0, & \text{otherwise}. \end{cases} \tag{1.2}$$

Then observe that $g_n \to g = 0$ (equi-stat) on $[0,1]$, however (g_n) is not statistically (or ordinary) uniform convergent to the function $g = 0$ on the interval $[0,1]$ (see, for details, [2]).

Now let $\{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. In this case, Korovkin [12] first noticed the necessary and sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test function e_i defined by $e_i(x) = x^i$ $(i = 0,1,2)$. Later many researchers investigate these conditions for various operators defined on different spaces. In recent years, some matrix summability methods have been used in the approximation theory. Although some operators, such as interpolation operators of Hermite–Fejer [3], do not converge at points of simple discontinuity, the matrix summability method of Cesàro-type are strong enough to correct the lack of convergence [4]. Furthermore, uniform statistical convergence in Definition 1.3, which is a regular (non-matrix) summability transformation, has been used in the Korovkin-type approximation theory [7–9,11]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem.

With the above terminology, our primary interest in the present paper is to obtain a Korovkin-type approximation theorem by means of the concept of equi-statistical convergence in Definition 1.2. Also, by considering Lemma 1.1 and the above example, we will construct a sequence of positive linear operators such that while our new results work, their classical and also uniform statistical cases do not work. We also compute the rates of equi-statistical convergence of sequences of positive linear operators. Finally, we obtain a Voronovskaya-type theorem in the equi-statistical sense for a sequence of positive linear operators constructed by means of the Bernstein polynomials.

2. A Korovkin-type approximation theorem

In this section we prove the following Korovkin-type approximation theorem by means of equi-statistical convergence.

Theorem 2.1. *Let X be a compact subset of the real numbers, and let $\{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. Then, for all $f \in C(X)$,*

$$L_n(f) \to f \quad \text{(equi-stat) on } X \tag{2.1}$$

if and only if

$$L_n(e_i) \to e_i \quad \text{(equi-stat) on } X \quad \text{with} \quad e_i(x) = x^i,\ i = 0, 1, 2. \tag{2.2}$$

Proof. Since each $e_i \in C(X)$, $i = 0, 1, 2$, the implication (2.1) $\Rightarrow$ (2.2) is obvious. Assume now that (2.2) holds. Let $f \in C(X)$ and $x \in X$ be fixed. By the continuity of f at the point x, we may write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in X$ satisfying $|y - x| < \delta$. Since

$$\left|f(y) - f(x)\right| = \left|f(y) - f(x)\right|\chi_{X_\delta}(y) + \left|f(y) - f(x)\right|\chi_{X \setminus X_\delta}(y),$$

where $X_\delta = [x - \delta, x + \delta] \cap X$ and χ_A denotes the characteristic function of the set A. Then we have

$$\left|f(y) - f(x)\right| \leqslant \varepsilon + 2M\frac{(y-x)^2}{\delta^2}$$

for all $y \in X$, where $M := \|f\|_{C(X)}$. By positivity and linearity of the operators L_n, we conclude that

$$\left|L_n(f;x) - f(x)\right| \leqslant \varepsilon + (\varepsilon + M)\left|L_n(e_0;x) - e_0(x)\right| + \frac{2M}{\delta^2}L_n\big((y-x)^2;x\big).$$

So we get

$$\begin{aligned}\left|L_n(f;x) - f(x)\right| \leqslant \varepsilon &+ \left(\varepsilon + M + \frac{2Mx^2}{\delta^2}\right)\left|L_n(e_0;x) - e_0(x)\right| \\ &+ \frac{4Mx}{\delta^2}\left|L_n(e_1;x) - e_1(x)\right| + \frac{2M}{\delta^2}\left|L_n(e_2;x) - e_2(x)\right|,\end{aligned}$$

which implies that

$$\left|L_n(f;x) - f(x)\right| \leqslant \varepsilon + K\sum_{i=0}^{2}\left|L_n(e_i;x) - e_i(x)\right|, \tag{2.3}$$

where

$$K := \varepsilon + M + \frac{2M}{\delta^2}\big(\|e_2\|_{C(X)} + 2\|e_1\|_{C(X)} + 1\big).$$

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Then, for each $i = 0, 1, 2$, setting

$$\Psi_n(x,r) := \left|\left\{k \leqslant n\colon \left|L_k(f;x) - f(x)\right| \geqslant r\right\}\right|$$

and

$$\Psi_{i,n}(x,r) := \left|\left\{k \leqslant n\colon \left|L_k(e_i;x) - e_i(x)\right| \geqslant \frac{r-\varepsilon}{3K}\right\}\right| \quad (i = 0, 1, 2),$$

it follows from (2.3) that

$$\Psi_n(x,r) \leqslant \sum_{i=0}^{2}\Psi_{i,n}(x,r),$$

which gives

$$\frac{\|\Psi_n(\cdot,r)\|_{C(X)}}{n} \leqslant \sum_{i=0}^{2}\left(\frac{\|\Psi_{i,n}(\cdot,r)\|_{C(X)}}{n}\right). \tag{2.4}$$

Then using the hypothesis (2.2) and considering Definition 1.2, the right-hand side of (2.4) tends to zero as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \frac{\|\Psi_n(\cdot, r)\|_{C(X)}}{n} = 0 \quad \text{for every } r > 0,$$

whence the result. $\square$

Now let $X = [0, 1]$ and consider the classical Bernstein polynomials $B_n(f; x)$ on $C[0, 1]$. Using these polynomials, we introduce the following positive linear operators on $C[0, 1]$:

$$D_n(f; x) = \bigl(1 + g_n(x)\bigr)B_n(f; x), \quad x \in [0, 1] \text{ and } f \in C[0, 1], \tag{2.5}$$

where $g_n(x)$ is given by (1.2). Then, observe that

$$D_n(e_0; x) = \bigl(1 + g_n(x)\bigr)e_0(x),$$
$$D_n(e_1; x) = \bigl(1 + g_n(x)\bigr)e_1(x),$$
$$D_n(e_2; x) = \bigl(1 + g_n(x)\bigr)\left[e_2(x) + \frac{x(1 - x)}{n}\right].$$

Since $g_n \to g = 0$ (equi-stat) on $[0, 1]$, we conclude that

$$D_n(e_i) \to e_i \quad \text{(equi-stat) on } [0, 1] \quad \text{for each } i = 0, 1, 2.$$

So, by Theorem 2.1, we immediately see that

$$D_n(f) \to f \quad \text{(equi-stat) on } [0, 1] \quad \text{for all } f \in C[0, 1].$$

However, since (g_n) is not statistically uniform convergent to the function $g = 0$ on the interval $[0, 1]$, we can say that Theorem 1 of [11] does not work for our operators defined by (2.5). Furthermore, since (g_n) is not uniformly convergent (in the ordinary sense) to the function $g = 0$ on $[0, 1]$, the classical Korovkin theorem does not work either. Therefore, this application clearly shows that our Theorem 2.1 is a non-trivial generalization of the classical and the statistical cases of the Korovkin-type results introduced in [12] and [11], respectively.

3. Rates of equi-statistical convergence in Theorem 2.1

In this section we study the rates of equi-statistical convergence of a sequence of positive linear operators defined on $C(X)$ with the help of modulus of continuity.

We now present the following definition.

Definition 3.1. A sequence (f_n) is equi-statistical convergent to a function f with the rate of $\beta \in (0, 1)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\Psi_n(x, \varepsilon)}{n^{1-\beta}} = 0$$

uniformly with respect to $x \in X$, or equivalently, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\|\Psi_n(\cdot, \varepsilon)\|_{C(X)}}{n^{1-\beta}} = 0,$$

where $\Psi_n(x, \varepsilon)$ is the same as in (1.1). In this case, it is denoted by

$$f_n - f = o\bigl(n^{-\beta}\bigr) \quad \text{(equi-stat) on } X.$$

Then we first need the following lemma to get the rates of equi-statistical convergence in Theorem 2.1 by using Definition 3.1.

Lemma 3.1. *Let (f_n) and (g_n) be function sequences belonging to $C(X)$. Assume that $f_n - f = o(n^{-\beta_0})$ (equi-stat) on X and $g_n - g = o(n^{-\beta_1})$ (equi-stat) on X. Let $\beta = \min\{\beta_0, \beta_1\}$. Then the following statements hold:*

(i) $(f_n + g_n) - (f + g) = o(n^{-\beta})$ *(equi-stat) on* X,
(ii) $(f_n - f)(g_n - g) = o(n^{-\beta})$ *(equi-stat) on* X,
(iii) $\lambda(f_n - f) = o(n^{-\beta_0})$ *(equi-stat) on* X *for any real number* λ,
(iv) $\sqrt{|f_n - f|} = o(n^{-\beta_0})$ *(equi-stat) on* X.

Proof. (i) Assume that $f_n - f = o(n^{-\beta_0})$ (equi-stat) on X and that $g_n - g = o(n^{-\beta_1})$ (equi-stat) on X. Also, for $\varepsilon > 0$ and $x \in X$ define

$$\Psi_n(x,\varepsilon) := \left|\left\{k \leqslant n\colon \left|(f_k + g_k)(x) - (f+g)(x)\right| \geqslant \varepsilon\right\}\right|,$$
$$\Psi_{0,n}(x,\varepsilon) := \left|\left\{k \leqslant n\colon \left|f_k(x) - f(x)\right| \geqslant \frac{\varepsilon}{2}\right\}\right|,$$
$$\Psi_{1,n}(x,\varepsilon) := \left|\left\{k \leqslant n\colon \left|g_k(x) - g(x)\right| \geqslant \frac{\varepsilon}{2}\right\}\right|.$$

Then observe that

$$\frac{\Psi_n(x,\varepsilon)}{n^{1-\beta}} \leqslant \frac{\Psi_{0,n}(x,\varepsilon) + \Psi_{1,n}(x,\varepsilon)}{n^{1-\beta}}. \tag{3.1}$$

Since $\beta = \min\{\beta_0, \beta_1\}$, by (3.1), we get

$$\frac{\Psi_n(x,\varepsilon)}{n^{1-\beta}} \leqslant \frac{\Psi_{0,n}(x,\varepsilon)}{n^{1-\beta_0}} + \frac{\Psi_{1,n}(x,\varepsilon)}{n^{1-\beta_1}},$$

and hence

$$\frac{\|\Psi_n(\cdot,\varepsilon)\|_{C(X)}}{n^{1-\beta}} \leqslant \frac{\|\Psi_{0,n}(\cdot,\varepsilon)\|_{C(X)}}{n^{1-\beta_0}} + \frac{\|\Psi_{1,n}(\cdot,\varepsilon)\|_{C(X)}}{n^{1-\beta_1}}. \tag{3.2}$$

Now by taking limit as $n \to \infty$ in (3.2) and using the hypotheses, we conclude that

$$\lim_{n\to\infty} \frac{\|\Psi_n(\cdot,\varepsilon)\|_{C(X)}}{n^{1-\beta}} = 0,$$

which completes the proof of (i). Since the proofs of (ii)–(iv) are similar, we omit them. □

On the other hand, we recall that the modulus of continuity of a function $f \in C(X)$ is defined by

$$w(f,\delta) = \sup_{|y-x|\leqslant\delta,\, x,y\in X} \left|f(y) - f(x)\right| \quad (\delta > 0).$$

Then we have the following result.

Theorem 3.2. *Let X be a compact subset of the real numbers, and let $\{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. Assume that the following conditions hold*:

(a) $L_n(e_0) - e_0 = o(n^{-\beta_0})$ *(equi-stat) on* X,
(b) $w(f,\delta_n) = o(n^{-\beta_1})$ *(equi-stat) on* X, *where* $\delta_n(x) = \sqrt{L_n(\varphi^2; x)}$ *with* $\varphi(y) = (y - x)$.

Then we have, for all $f \in C(X)$,

$$L_n(f) - f = o\big(n^{-\beta}\big) \quad \textit{(equi-stat) on } X,$$

where $\beta = \min\{\beta_0, \beta_1\}$.

Proof. Let $f \in C(X)$ and $x \in X$. Then it is well known that,

$$\left|L_n(f;x) - f(x)\right| \leqslant M\left|L_n(e_0;x) - e_0(x)\right| + \left(L_n(e_0;x) + \sqrt{L_n(e_0;x)}\right)w(f,\delta_n),$$

where $M = \|f\|_{C(X)}$ (see, for instance, [1,5]). This yields that

$$\left|L_n(f;x)-f(x)\right| \leqslant M\left|L_n(e_0;x)-e_0(x)\right| + 2w(f,\delta_n) + w(f,\delta_n)\left|L_n(e_0;x)-e_0(x)\right| + w(f,\delta_n)\sqrt{\left|L_n(e_0;x)-e_0(x)\right|}.$$

Now considering the above inequality, the hypotheses (a), (b), and Lemma 3.1, the proof is completed at once. □

Remark. Replace the conditions (a) and (b) in Theorem 3.2 by

$$L_n(e_i)-e_i = o\left(n^{-\beta_i}\right) \quad \text{(equi-stat) on } X \quad \text{for } i=0,1,2. \tag{3.3}$$

Since

$$L_n\left(\varphi^2;x\right) = L_n(e_2;x) - 2xL_n(e_1;x) + x^2 L_n(e_0;x),$$

we may write that

$$L_n\left(\varphi^2;x\right) \leqslant K\sum_{i=0}^{2}\left|L_n(e_i;x)-e_i(x)\right|, \tag{3.4}$$

where $K = 1 + 2\|e_1\|_{C(X)} + \|e_2\|_{C(X)}$. Now it follows from (3.3), (3.4) and Lemma 3.1 that

$$\delta_n = \sqrt{L_n\left(\varphi^2\right)} = o\left(n^{-\gamma}\right) \quad \text{(equi-stat) on } X,$$

where $\gamma = \min\{\beta_0, \beta_1, \beta_2\}$. So, it yields that

$$w(f,\delta_n) = o\left(n^{-\gamma}\right) \quad \text{(equi-stat) on } X. \tag{3.5}$$

Using (3.5) in Theorem 3.2 we immediately see, for all $f \in C(X)$, that

$$L_n(f) - f = o\left(n^{-\gamma}\right) \quad \text{(equi-stat) on } X.$$

Therefore, if we use the condition (3.3) in Theorem 3.2 instead of (a) and (b), then we obtain the rates of equi-statistical convergence of the sequence of positive linear operators in Theorem 2.1.

4. A Voronovskaya-type theorem

In this section, we will show that the positive linear operators D_n defined by (2.5) satisfy a Voronovskaya-type property in the equi-statistical sense. We first need the following lemma.

Lemma 4.1. *Let $x \in [0,1]$ and $\varphi(y) := y - x$. Then we have*

$$n^2 D_n\left(\varphi^4\right) \to 3e_2(e_2 - 2e_1 + e_0) \quad \textit{(equi-stat) on } [0,1].$$

Proof. After some simple calculations, we may write form (2.5) that

$$n^2 D_n\left(\varphi^4;x\right) = \left(1+g_n(x)\right)\left[3x^4 - 6x^3 + 3x^2 + \frac{-6x^4 + 12x^3 - 7x^2 + x}{n}\right].$$

Then we get

$$\left|n^2 D_n\left(\varphi^4;x\right) - 3e_2(x)\left(e_2(x) - 2e_1(x) + e_0(x)\right)\right| \leqslant 12g_n(x) + \frac{26(1+g_n(x))}{n} \tag{4.1}$$

for every $x \in [0,1]$. By (1.2), it is clear that

$$12g_n + \frac{26(1+g_n)}{n} \to 0 \quad \text{(equi-stat) on } [0,1]. \tag{4.2}$$

Now combining (4.1) and (4.2) we easily see that

$$n^2 D_n\left(\varphi^4\right) \to 3e_2(e_2 - 2e_1 + e_0) \quad \text{(equi-stat) on } [0,1],$$

which completes the proof. □

Then we obtain the following Voronovskaya-type result for the operators D_n given by (2.5).

Theorem 4.2. *For every $f \in C[0,1]$ such that $f', f'' \in C[0,1]$, we have*

$$n\{D_n f - f\} \to \frac{e_1 - e_2}{2} f'' \quad (\textit{equi-stat}) \textit{ on } [0,1].$$

Proof. Let $f, f', f'' \in C[0,1]$ and $x \in [0,1]$. Define

$$\zeta_x(y) = \begin{cases} \frac{f(y)-f(x)-(y-x)f'(x)-\frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2}, & \text{if } y \neq x, \\ 0, & \text{if } y = x. \end{cases}$$

Then by assumption we have $\zeta_x(x) = 0$ and the function ζ_x belongs to $C[0,1]$. Hence, by Taylor's theorem we get

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2} f''(x) + (y-x)^2 \zeta_x(y).$$

Now applying the operators D_n given by (2.5) on the both sides of the above equality, we conclude that

$$\begin{aligned} D_n(f;x) - f(x) &= f(x)g_n(x) + f'(x)D_n(\varphi;x) + \frac{f''(x)}{2} D_n(\varphi^2;x) + D_n(\varphi^2\zeta_x;x) \\ &= f(x)g_n(x) + \frac{x(1-x)f''(x)}{2n}\big(1+g_n(x)\big) + D_n(\varphi^2\zeta_x;x), \end{aligned}$$

which yields

$$\left| n\big(D_n(f;x) - f(x)\big) - \frac{e_1(x)-e_2(x)}{2} f''(x) \right| \leqslant M n g_n(x) + n\big|D_n(\varphi^2\zeta_x;x)\big|, \tag{4.3}$$

where $\varphi(y) = y - x$ and $M = \|f\|_{C[0,1]} + \|f''\|_{C[0,1]}$. If we apply the Cauchy–Schwarz inequality for the second term on the right-hand side of (4.3), then we may write that

$$n\big|D_n(\varphi^2\zeta_x;x)\big| \leqslant \big(n^2 D_n(\varphi^4;x)\big)^{\frac{1}{2}} \big(D_n(\zeta_x^2;x)\big)^{\frac{1}{2}}. \tag{4.4}$$

Let $\eta_x(y) := \zeta_x^2(y)$. In this case, observe that $\eta_x(x) = 0$ and $\eta_x(\cdot) \in C[0,1]$. Then it follows from Theorem 2.1 that

$$D_n(\eta_x) \to 0 \quad \text{(equi-stat) on } [0,1]. \tag{4.5}$$

Now considering (4.4) and (4.5), and also using Lemma 4.1, we immediately see that

$$nD_n(\varphi^2\zeta_x;x) \to 0 \quad \text{(equi-stat) on } [0,1]. \tag{4.6}$$

For a given $\varepsilon > 0$, define

$$\begin{aligned} \Psi_n(x,\varepsilon) &:= \left| \left\{ k \leqslant n \colon \left| k\big(D_k(f;x) - f(x)\big) - \frac{e_1(x)-e_2(x)}{2} f''(x) \right| \geqslant \varepsilon \right\} \right|, \\ \Psi_{1,n}(x,\varepsilon) &:= \left| \left\{ k \leqslant n \colon \big|k g_k(x)\big| \geqslant \frac{\varepsilon}{2M} \right\} \right|, \\ \Psi_{2,n}(x,\varepsilon) &:= \left| \left\{ k \leqslant n \colon \big|k D_k(\varphi^2\zeta_x;x)\big| \geqslant \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

Then it follows from (4.3) that

$$\frac{\Psi_n(x,\varepsilon)}{n} \leqslant \frac{\Psi_{1,n}(x,\varepsilon)}{n} + \frac{\Psi_{2,n}(x,\varepsilon)}{n},$$

which yields

$$\frac{\|\Psi_n(\cdot,\varepsilon)\|_{C[0,1]}}{n} \leqslant \frac{\|\Psi_{1,n}(\cdot,\varepsilon)\|_{C[0,1]}}{n} + \frac{\|\Psi_{2,n}(\cdot,\varepsilon)\|_{C[0,1]}}{n}. \tag{4.7}$$

Also, by the definition of g_n, observe that

$$ng_n \to g = 0 \quad \text{(equi-stat) on } [0, 1]. \tag{4.8}$$

Now, by (4.6) and (4.8), the right-hand side of (4.7) tends to zero as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \frac{\|\Psi_n(\cdot, \varepsilon)\|_{C[0,1]}}{n} = 0$$

whence the result. $\square$

Finally, notice that our operators D_n defined by (2.5) do not satisfy the Voronovskaya-type property in the usual sense since the function sequence (g_n) given by (1.2) is not uniformly convergent to the function $g = 0$ on the interval [0, 1].

References

[1] F. Altomare, M. Campiti, Korovkin-Type Approximation Theory and Its Applications, Walter de Gruyter Publ., Berlin, 1994.
[2] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007) 715–729.
[3] R. Bojanic, F. Cheng, Estimates for the rate of approximation of functions of bounded variation by Hermite–Fejer polynomials, Proc. Conference Canadian Math. Soc. 3 (1983) 5–17.
[4] R. Bojanic, M.K. Khan, Summability of Hermite–Fejer interpolation for functions of bounded variation, J. Natur. Sci. Math. 32 (1992) 5–10.
[5] R.A. Devore, The Approximation of Continuous Functions by Positive Linear Operators, Lecture Notes in Math., vol. 293, Springer-Verlag, Berlin, 1972.
[6] O. Duman, C. Orhan, μ-Statistically convergent function sequences, Czechoslovak Math. J. 54 (2004) 413–422.
[7] O. Duman, M.K. Khan, C. Orhan, A-statistical convergence of approximating operators, Math. Inequal. Appl. 6 (2003) 689–699.
[8] E. Erkuş, O. Duman, A Korovkin type approximation theorem in statistical sense, Studia Sci. Math. Hungar. 43 (2006) 285–294.
[9] E. Erkuş, O. Duman, H.M. Srivastava, Statistical approximation of certain positive linear operators constructed by means of the Chan–Chyan–Srivastava polynomials, Appl. Math. Comput. 182 (2006) 213–222.
[10] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[11] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002) 129–138.
[12] P.P. Korovkin, Linear Operators and Approximation Theory, Hindustan Publ. Co., Delhi, 1960.